Combinatorial Networks
Week 5, Thursday

- **Fact.** If $H_1 \subseteq H_2$, then $ex(n, H_1) \leq ex(n, H_2)$.
  Proof. For any H_1-free graph is also H_2-free. ∎

- **Theorem.** For $s \geq t \geq 2$, $ex(n, K_{s,t}) \leq Cs^{\frac{1}{t}} n^{2-\frac{1}{t}}$ for some constant C.
  Proof. Consider any $K_{s,t}$-free graph G on n vertices, we count the number N of copies (v, T), where $T \subset N(v)$ and $|T| = t$, then $N = \sum_{v \in V(G)} \binom{d(v)}{t}$.
  Since G has no $K_{s,t}$, we have $N \leq (s-1)\binom{n}{t}$. Otherwise, there is a $T \subset V(G)$ of size t such that there are at least s many (v, t) pairs. Thus

$$(s-1)\binom{n}{t} \geq \sum_{v \in V(G)} \binom{d(v)}{t} \geq n \binom{\frac{\sum d(v)}{n}}{t}$$

  (By Jensen's inequality as $f(x) = \binom{x}{t}$ convex), so

$$ex(G) \leq \frac{1}{2}(2S-1)^{\frac{1}{t}} n^{2-\frac{1}{t}}.$$

  ∎

- **Theorem.** (Erdös-Stone-Simonovits) For any graph H with $\chi(H) = k+1$, $ex(n, H) = (1 - \frac{1}{k} + o_H(1))\frac{n^2}{2}$.
  Proof. Lower bound: $T_k(n)$ is k-colorable, so $H \not\subseteq T_k(n)$.
  Upper bound: By induction on $\chi(H) = k+1$.
  Base case, when $k = 1$, $\chi(H) = 2$. Let $t = |V(H)|$, then $H \subseteq K_{t,t}$, by fact,

$$ex(n, H) \leq ex(n, K_{t,t}) \leq Ct^{\frac{1}{t}} n^{2-\frac{1}{t}} = o(n^2).$$

  Inductive step: assume it is true for all $\widetilde{H}$ with $\chi(\widetilde{H}) \leq k$. Let H with $\chi(H) = k+1$, consider H-free G on n vertices, let $t = |V(H)|$.(so k, t are fixed)
  It suffices to prove: For any $\varepsilon > 0$ and sufficiently large n, any H-free graph G on n vertices has $ex(G) \leq (1 - \frac{1}{k} + \varepsilon)\frac{n^2}{2}$.
  In fact, we prove a stronger result: For any $\varepsilon > 0$ and sufficiently large n, any G with n vertices and $m \geq (1 - \frac{1}{k} + \varepsilon)\frac{n^2}{2}$ contains $T_{k+1}((k+1)t)$. ($T_{k+1}((k+1)t) \supseteq H$)
  Claim.: There is $\widetilde{G} \subset G$ with $|\widetilde{G}| = \widetilde{n}$ such that
  -any $v \in V(\widetilde{G})$ has degree at least $(1 - \frac{1}{k} + \frac{2\varepsilon}{3})\widetilde{n}$.
  -$\widetilde{n} \geq o(\sqrt{\varepsilon})n$.
  Proof of claim. Let $G_0 := G$, for G_i, if there is $v \in V(G)$ such that $d_{G_i}(v) < (1 - \frac{1}{k} + \frac{2\varepsilon}{3})n_i$, where $n_i = |V(G_i)|$, then denote $G_{i+1} = G_i - v$, otherwise we stop at G_j. In this process, the number of deleted edge$\leq (1 - \frac{1}{k} + \frac{2\varepsilon}{3})[n + (n-1) + ... + (n-j)] \leq (1 - \frac{1}{k} + \frac{2\varepsilon}{3})\frac{n^2}{2} < e(G)$.
  So G_j is not empty and

$$\begin{aligned} e(G_j) &\geq e(G) - \#\text{deleted edges} \\ &\geq (1 - \frac{1}{k} + \varepsilon)\frac{n^2}{2} - (1 - \frac{1}{k} + \frac{2\varepsilon}{3})\frac{n^2}{2} \\ &= \frac{\varepsilon}{6} n^2 \end{aligned}$$

But $e(G_j) \le \frac{n_j^2}{2}$, then $\frac{n_j^2}{2} \ge \frac{\varepsilon}{6}n^2$, so $n_j \ge o(\sqrt{\varepsilon})n$. ∎

From now on, we will view G_j as the new "G" and treat n_j as the new "n", $\delta(G) \ge (1-\frac{1}{k}+\frac{2\varepsilon}{3})n$. Let

$$R = K_{\underbrace{s, s, ..., s}_{k}}$$

where $s = \lceil \frac{t}{\varepsilon} \rceil$ is fixed, so $\chi(R) = k$.

By induction, $e(G) \ge (1-\frac{1}{k}+\frac{2\varepsilon}{3})\frac{n^2}{2} > (1-\frac{1}{k-1}+o_R(1))\frac{n^2}{2} = ex(n, R)$, then G has a copy of R.

Let $R = K(B_1, B_2, ..., B_k)$, where $|B_i| = s$, let $U = V(G) - \bigcup_{i=1}^{k} B_i$, let $W = \{u \in U : |N(u) \bigcap B_i| \ge t$ for all $i = 1, 2, ..., k\}$, let $\widetilde{e}$ be the number of missing edges between R and U, we have

$$\widetilde{e} \le |\bigcup_{i=1}^{k} B_i|(\frac{1}{k}-\frac{2\varepsilon}{3})n = ks(\frac{1}{k}-\frac{2\varepsilon}{3})n$$

and

$$\widetilde{e} \ge |U \setminus W|(s-t) \ge (|U|-|W|)s(1-\varepsilon)$$

Thus

$$ks(\frac{1}{k}-\frac{2\varepsilon}{3})n \ge (|U|-|W|)s(1-\varepsilon)$$

But $|U| = n - ks$, so

$$|W|(1-\varepsilon) \ge n\varepsilon(\frac{2k}{3}-1) - ks \ge \frac{\varepsilon}{3}n - ks \ge \frac{\varepsilon}{6}n$$

We have

$$|W| \ge \frac{\varepsilon}{10}n > \binom{s}{t}^k (t-1).$$

By pigeonhole's principle, there is $A_i \subseteq B_i$ of size t and $A_{k+1} \subseteq W$ such that $|A_i| = t$ for $1 \le i \le k+1$ and $\bigcup_{i=1}^{k+1} A_i$ is

$$K_{\underbrace{t, t, ..., t}_{k+1}} = T_{k+1}((k+1)t) \supseteq H.$$

∎

To see the last step (pigeonhole's principle), consider the following: For any $A_i \subseteq B_i$ of size t, define $W(A_1, ..., A_k) = \{u \in W : N(u) \supseteq \bigcup_{i=1} k + 1A_i\}$, so W can be partitioned into $\binom{s}{t}^k$ $W(A_1, ..., A_k)$.